Quantum kinetic equations in curved space–time and thermal equilibrium

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Collisionless quantum kinetic equations are considered in curved space–time with symmetries. It is shown that, if the Killing vector exists, then the quantum-corrected Vlasov equation has a solution of the form of a local functional of the solution of the classical Vlasov equation. Curvature corrections of the energy–momentum tensor are obtained, which are the moments of the classical distribution function and its derivatives.

One of the most effective methods in quantum kinetic field theory is the method of Wigner functions. This method in curved space–time was developed in refs. [1–4] for deriving the Vlasov equation [1] and its quantum corrections caused by the inhomogeneity of the external gravitational and gauge fields [2–4] and the interaction of the quantum field's spin or isospin with the external fields [4]. In this paper we consider some properties of the equations for the Wigner function of a scalar field and their solution in curved space–time with symmetries. Let us use the equations up to the second order in terms of the Planck constant $\hbar$ in the following form, which was obtained in refs. [2,4],

$$(m^2-p^\alpha p_\alpha)f(x,p)+\hbar^2\Big(\tfrac{1}{4}\check{\nabla}^\alpha\check{\nabla}_\alpha+(\tfrac{1}{3}-\xi)R$$
$$+\tfrac{1}{12}R_{\alpha\mu\beta\nu}p^\mu p^\nu\frac{\partial^2}{\partial p_\alpha\partial p_\beta}+\tfrac{1}{4}R_{\mu\nu}p^\mu\frac{\partial}{\partial p_\nu}\Big)f(x,p)$$
$$=0\,, \tag{1}$$

$$p^\mu\check{\nabla}_\mu f(x,p)-\hbar^2\Big(\tfrac{1}{12}R_{\mu\nu}\frac{\partial}{\partial p_\mu}\check{\nabla}^\nu$$
$$+\tfrac{1}{6}R_{\alpha\mu\beta\nu}p^\mu\frac{\partial^2}{\partial p_\alpha\partial p_\beta}\check{\nabla}^\nu-\tfrac{1}{24}R_{\alpha\mu\beta\nu;\gamma}p^\mu p^\nu\frac{\partial^3}{\partial p_\alpha\partial p_\beta\partial p_\gamma}$$
$$-\tfrac{1}{24}R_{\alpha\beta;\mu}p^\mu\frac{\partial^2}{\partial p_\alpha\partial p_\beta}+\tfrac{1}{2}(\xi-\tfrac{1}{4})R_{;\mu}\frac{\partial}{\partial p_\mu}\Big)f(x,p)$$
$$=0\,. \tag{2}$$

Eq. (1) gives the quantum and the curvature corrections to the mass shell, and eq. (2) is the quantum Liouville–Vlasov equation. Here $\check{\nabla}_\mu$ is the Cartan covariant derivation operator in cotangential stratification:

$$\check{\nabla}_\mu f(x,p)=\partial_\mu f(x,p)+\Gamma^\beta_{\mu\alpha}p_\beta\frac{\partial}{\partial p_\alpha}f(x,p)\,. \tag{3}$$

As it was noted in ref. [2], eqs. (1) and (2) may be seen as the equations for the eigenvalue zero of the two non-commutating operators. Therefore, it is necessary to consider the integration conditions of eqs. (1) and (2). To avoid this difficulty let us use an ansatz for the function $f(x,p)$ differing somewhat from the one of ref. [3]:

$$f(x,p)=\sum_{n=0}^{\infty}F_n(x,p)\delta^{(n)}(m^2-p^\alpha p_\alpha)\,. \tag{4}$$